

ON A CUBIC MOMENT OF HARDY'S FUNCTION WITH A SHIFT

ALEKSANDAR IVIĆ

ABSTRACT. An asymptotic formula for

$$\int_{T/2}^T Z^2(t)Z(t+U) \, dt \quad (0 < U = U(T) \leq T^{1/2-\varepsilon})$$

is derived, where

$$Z(t) := \zeta\left(\frac{1}{2} + it\right)\left(\chi\left(\frac{1}{2} + it\right)\right)^{-1/2} \quad (t \in \mathbb{R}), \quad \zeta(s) = \chi(s)\zeta(1-s)$$

is Hardy's function. The cubic moment of $Z(t)$ is also discussed, and a mean value result is presented which supports the author's conjecture that

$$\int_1^T Z^3(t) \, dt = O_\varepsilon(T^{3/4+\varepsilon}).$$

1. INTRODUCTION

Let $\zeta(s)$ denote the Riemann zeta-function, and as usual let us define *Hardy's function* $Z(t)$ as

$$Z(t) := \zeta\left(\frac{1}{2} + it\right)\left(\chi\left(\frac{1}{2} + it\right)\right)^{-1/2} \quad (t \in \mathbb{R}), \quad \zeta(s) = \chi(s)\zeta(1-s).$$

We have (see e.g., [5, Chapter 1] and [6]), with $s = \sigma + it$ a complex variable,

$$(1.1) \quad \chi(s) = 2^s \pi^{s-1} \sin\left(\frac{1}{2}\pi s\right) \Gamma(1-s) = \left(\frac{2\pi}{t}\right)^{\sigma+it-1/2} e^{i(t+\pi/4)} \left(1 + O\left(\frac{1}{t}\right)\right)$$

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if $t \geq t_0$ (> 0). In fact, one can obtain a full asymptotic expansion for $\chi(s)$ from the classical Stirling formula for the gamma-function.

The function $Z(t)$ (see [8] for an extensive account) is real-valued, even, smooth, and as $|Z(t)| = |\zeta(\frac{1}{2} + it)|$, its zeros coincide with the zeros of $\zeta(s)$ on the “critical line” $\Re s = \frac{1}{2}$. It is indispensable in the study of the zeros of $\zeta(s)$ on the critical line.

The moments of $Z(t)$ are of interest, but since $|Z(t)| = |\zeta(\frac{1}{2} + it)|$, only odd moments represent a novelty. M. Korolev [14] and M. Jutila [10], [11] showed independently that

$$\int_0^T Z(t) dt = O(T^{1/4}), \quad \int_0^T Z(t) dt = \Omega_{\pm}(T^{1/4}),$$

thus establishing (up to the value of the numerical constants which are involved) the true order of the integral in question. This improves on the author’s bound $O_{\varepsilon}(T^{1/4+\varepsilon})$, obtained in [7].

However, higher odd moments of $Z(t)$, where one also expects a lot of cancellations due to the oscillatory nature of the function $Z(t)$, remain a mystery.

In [8], equation (11.9), an explicit formula for the cubic moment of $Z(t)$ was presented. This is

$$(1.2) \quad \int_T^{2T} Z^3(t) dt = 2\pi \sqrt{\frac{2}{3}} \sum_{(\frac{T}{2\pi})^{3/2} \leq n \leq (\frac{T}{\pi})^{3/2}} d_3(n) n^{-\frac{1}{6}} \cos\left(3\pi n^{\frac{2}{3}} + \frac{1}{8}\pi\right) + O_{\varepsilon}(T^{3/4+\varepsilon}),$$

where as usual $d_3(n)$ is the divisor function

$$d_3(n) = \sum_{k\ell m=n} 1 \quad (k, \ell, m, n \in \mathbb{N}),$$

generated by $\zeta^3(s)$ for $\Re s > 1$.

Remark 1. Here and later ε denotes positive constants which are arbitrarily small, but are not necessarily the same ones at each occurrence, while $a(x) \ll_{\varepsilon} b(x)$ (same as $a(x) = O_{\varepsilon}(b(x))$) means that the $|a(x)| \leq Cb(x)$ for some $C = C(\varepsilon) > 0$, $x \geq x_0$. By $a \asymp b$ we mean that $a \ll b \ll a$. The symbol $f(x) = \Omega_{\pm}(g(x))$ means that both $\limsup_{x \rightarrow \infty} f(x)/g(x) > 0$ and $\liminf_{x \rightarrow \infty} f(x)/g(x) < 0$ holds.

The motivation for this investigation is the following problem, first posed by the author in Oberwolfach 2003 during the conference “Elementary and Analytic Number Theory”: Does there exist a constant c with $0 < c < 1$ such that

$$(1.3) \quad \int_1^T Z^3(t) dt = O(T^c)?$$

This is equation (11.8) of [8]. So far obtaining (1.3) with any $c < 1$ is an open problem, but if one considers the cubic moment of $|Z(t)|$, then it is known that

$$(1.4) \quad T(\log T)^{9/4} \ll \int_1^T |Z(t)|^3 dt = \int_1^T |\zeta(\tfrac{1}{2} + it)|^3 dt \ll T(\log T)^{9/4},$$

which establishes the true order of the integral in question. However, obtaining an asymptotic formula for this integral remains a difficult problem. The lower bound in (1.4) follows from general results of K. Ramachandra (see his monograph [15]), and the upper bound is a recent result of S. Bettin, V. Chandee and M. Radziwiłł [2].

A strong conjecture of the author is that

$$(1.5) \quad \int_1^T Z^3(t) dt = O_\varepsilon(T^{3/4+\varepsilon}).$$

Note that (1.5) would follow from (1.2) and the bound

$$(1.6) \quad \sum_{N < n \leq N' \leq 2N} d_3(n) e^{3\pi i n^{2/3}} \ll_\varepsilon N^{2/3+\varepsilon}.$$

It may be remarked that the exponential sum in (1.6) is “pure” in the sense that the function in the exponential does not depend on any parameter as, for example, the sum

$$\sum_{N < n \leq N' \leq 2N} n^{it} = \sum_{N < n \leq N' \leq 2N} e^{it \log n} \quad (1 \leq N \ll \sqrt{t}),$$

which appears in the approximation to $\zeta(\frac{1}{2} + it)$ (see e.g., Theorem 4.1 of [5]), depends on the parameter t . However, the difficulty in the estimation of the sum in (1.6) lies in the presence of the divisor function $d_3(n)$ which, in spite of its simple appearance, is quite difficult to deal with.

Remark 2. A natural question is to ask: What is the minimal value of c for which (1.3) holds ($c = 1 + \varepsilon$ is trivial)? In other words, to try to find an omega result for the integral in (1.3).

A related and interesting problem is to investigate integrals of $Z(t)$ with “shifts”, i.e., integrals where one (or more) factor $Z(t)$ is replaced by $Z(t + U)$. The parameter U , which does not depend on the variable of integration t , is supposed to be positive and $o(T)$ as $T \rightarrow \infty$, where T is the order of the range of integration.

Some results on such integrals already exist in the literature. For example, R.R. Hall [3] proved that, for $U = \alpha / \log T$, $\alpha \ll 1$, we have uniformly

$$(1.7) \quad \begin{aligned} \int_0^T Z(t)Z(t+U) dt &= \frac{\sin \alpha/2}{\alpha/2} T \log T + (2\gamma - 1 - 2\pi) T \cos \alpha/2 \\ &\quad + O\left(\frac{\alpha T}{\log T} + T^{1/2} \log T\right). \end{aligned}$$

In (1.7) $\gamma = -\Gamma'(1) = 0.5772157\dots$ is Euler's constant. M. Jutila [12] obtained recently an asymptotic formula for the left-hand side of (1.5) when $0 < U \ll T^{1/2}$.

S. Bettin [1] evaluated asymptotically a related integral, namely

$$\int_0^T \zeta(1/2 + U + it) \zeta(\tfrac{1}{2} - V - it) dt$$

under the condition that

$$U = U(T) \in \mathbb{C}, V = V(T) \in \mathbb{C}, \Re U \ll 1/\log T, \Re V \ll 1/\log T.$$

S. Shimomura [16] dealt with the quartic moment

$$(1.8) \quad \int_0^T Z^2(t) Z^2(t + U) dt,$$

under certain conditions on the real parameter U , such that $|U| + 1/\log T \rightarrow 0$ as $T \rightarrow \infty$. When $U = 0$, Shimomura's expression for (1.8) reduces to

$$(1.9) \quad \int_0^T |\zeta(\tfrac{1}{2} + it)|^4 dt = \int_0^T Z^4(t) dt = \frac{1}{2\pi^2} T \log^4 T + O(T \log^3 T).$$

The (weak) asymptotic formula (1.9) is a classical result of A.E. Ingham [4] of 1928. For the sharpest known asymptotic formula for the integral in (1.9), see the paper of Ivić–Motohashi [9].

2. STATEMENT OF RESULTS

The first purpose of this work is to investigate another integral with the shift of $Z(t)$, namely the cubic moment

$$(2.1) \quad \int_{T/2}^T Z^2(t) Z(t + U) dt$$

for a certain range of the “shift parameter” $U = U(T) > 0$. We shall prove the following

THEOREM 1. *For $0 < U = U(T) \leq T^{1/2-\varepsilon}$ we have, uniformly in U ,*

$$(2.2) \quad \int_{T/2}^T Z^2(t) Z(t + U) dt = O_\varepsilon(T^{3/4+\varepsilon}) + \\ + 2\pi \sqrt{\frac{2}{3}} \sum_{T_1 \leq n \leq T_0} h(n, U) n^{-1/6+iU/3} \exp(-3\pi i n^{2/3} - \pi i/8) \{1 + K(n, U)\}.$$

Here $(d(n))$ is the number of divisors of n)

$$(2.3) \quad h(n, U) := n^{-iU} \sum_{\delta|n} d(\delta) \delta^{iU}, \quad T_0 := \frac{T^{3/2}}{\sqrt{8\pi^3}}, \quad T_1 := \frac{\left(\frac{T}{2}\right)^{3/2}}{\sqrt{8\pi^3}},$$

$$(2.4) \quad K(n, U) := d_2 U^2 n^{-2/3} + \dots + d_k U^k n^{-2k/3} + O_k(U^{k+1} n^{-2(k+1)/3})$$

for any given integer $k \geq 2$, with effectively computable constants d_2, d_3, \dots .

Remark 3. When $U \rightarrow 0+$, the expressions in (2.2)–(2.4) tend to (1.2), since the integral is real and $\lim_{U \rightarrow 0+} h(n, U) = d_3(n)$. Thus Theorem 1 is a generalization of (1.2) for a relatively wide range of U .

Remark 4. By methods similar to those used in proving Theorem 1 one can deal also with the “conjugate” integral

$$\int_{T/2}^T Z^2(t+U) Z(t) dt \quad (0 < U \leq T^{1/2-\varepsilon}).$$

Remark 5. Analogously to (1.5) one may conjecture that, uniformly for $0 < U \leq T^{1/2-\varepsilon}$,

$$\int_0^T Z^2(t) Z(t+U) dt = O_\varepsilon(T^{3/4+\varepsilon}).$$

The second aim of this work is to demonstrate, although (1.5) and (1.6) seem at present intractable, that it is possible to show that the conjecture (1.6) holds in a certain mean value sense. To this end let

$$(2.5) \quad S(\alpha, N) := \sum_{N < n \leq N' \leq 2N} d_3(n) e^{\alpha i n^{2/3}} \quad (\alpha \in \mathbb{R}),$$

so that the exponential sum in (1.6) is $S(3\pi, N)$. Then we have

THEOREM 2. *Every finite interval $[A, B]$ ($0 < A < B$) contains at least one point C such that*

$$(2.6) \quad S(C, N) \ll_\varepsilon N^{2/3} \log^{9/2} N.$$

3. PROOF OF THE THEOREMS

We begin with the proof of Theorem 1. Suppose that $0 < U \leq T^{1/2-\varepsilon}$ and let

$$(3.1) \quad I := \int_{T/2}^T Z^2(t)Z(t+U) dt = \frac{1}{i} \int_{1/2+iT/2}^{1/2+iT} \zeta^2(s)\zeta(s+iU)(\chi^2(s)\chi(s+iU))^{-1/2} ds.$$

The procedure of writing a real-valued integral like a complex integral is fairly standard in analytic number theory. For example, see the proof of Theorem 7.4 in E.C. Titchmarsh's monograph [17] on $\zeta(s)$ and M. Jutila's work [12]. It allows one flexibility by suitably deforming the contour of integration in the complex plane. Incidentally, this method of proof is different from the proof of (1.2) in [5], which is based on the use of approximate functional equations.

Let

$$(3.2) \quad h(n, U) := n^{-iU} \sum_{\delta|n} d(\delta)\delta^{iU}, \quad N_0 = N_0(T, U) := \sqrt{\frac{T^2(T+U)}{8\pi^3}},$$

so that $h(n, U) \ll_{\varepsilon} n^{\varepsilon}$. We have

$$I = I_1 + I_2,$$

say, where

$$(3.3) \quad \begin{aligned} I_1 &:= \frac{1}{i} \int_{1/2+iT/2}^{1/2+iT} \sum_{n \leq N_0} h(n, U) n^{-s} (\chi^2(s)\chi(s+iU))^{-1/2} ds, \\ I_2 &:= \frac{1}{i} \int_{1/2+iT/2}^{1/2+iT} \left\{ \zeta^2(s)\zeta(s+iU) - \sum_{n \leq N_0} h(n, U) n^{-s} \right\} (\chi^2(s)\chi(s+iU))^{-1/2} ds, \end{aligned}$$

and we shall keep in mind that for $\Re s > 1$, since $d(n)$ is generated by $\zeta^2(s)$,

$$(3.4) \quad \sum_{n=1}^{\infty} h(n, U) n^{-s} = \sum_{k=1}^{\infty} \sum_{m=1}^{\infty} d(k) k^{iU} (km)^{-s} (km)^{-iU} = \zeta^2(s)\zeta(s+iU).$$

To be able to exploit (3.4) let henceforth $c = 1 + \varepsilon$. Then, by Cauchy's theorem,

$$I_2 = \frac{1}{i} \left\{ \int_{1/2+iT/2}^{c+iT/2} + \int_{c+iT}^{1/2+iT} + \int_{c+iT/2}^{c+iT} \right\} = \frac{1}{i} (J_1 + J_2 + J_3),$$

say. For $T/2 \leq t \leq T$, $\frac{1}{2} \leq \sigma \leq c$, $s = \sigma + it$, $0 < U \ll T^{1/2-\varepsilon}$ we have
(3.5)

$$\chi^2(s)\chi(s+iU) = e^{\frac{3\pi i}{4}} \left(\frac{2\pi}{t}\right)^{2\sigma+2it-1} \left(\frac{2\pi}{t+U}\right)^{\sigma+it+iU-\frac{1}{2}} e^{3it+iU} \left\{1 + O\left(\frac{1}{t}\right)\right\},$$

so that uniformly in U

$$\chi^2(s)\chi(s+iU) \asymp T^{3/2-3\sigma} \quad (t \asymp T).$$

This follows from the asymptotic formula (1.1), and actually the term $O(1/t)$ can be replaced by an asymptotic expansion. We also have the convexity bound (see e.g., Chapter 1 of [5])

$$(3.6) \quad \zeta(s) \ll_{\varepsilon} |t|^{(1-\sigma)/2+\varepsilon} \quad \left(\frac{1}{2} \leq \sigma \leq c\right).$$

Stronger bounds than (3.6) are available, but this bound is sufficient for our present purposes.

By using (3.5), (3.6) and the trivial bound

$$\sum_{n \leq N_0} h(n, U) n^{-s} \ll_{\varepsilon} N_0^{1-\sigma+\varepsilon} \ll_{\varepsilon} T^{(3-3\sigma)/2+\varepsilon} \quad \left(\frac{1}{2} \leq \sigma \leq c\right),$$

it follows readily that

$$(3.7) \quad J_1 + J_2 \ll_{\varepsilon} \int_{1/2}^c T^{(3-3\sigma)/2+\varepsilon} T^{(6\sigma-3)/4} d\sigma \ll_{\varepsilon} T^{3/4+\varepsilon}.$$

In J_3 we have $s = c + it$, $T/2 \leq t \leq T$. Hence

$$\begin{aligned} J_3 &= i \sum_{n > N_0} h(n, U) n^{-c} \int_{T/2}^T n^{-it} e^{-3\pi i/8} \left(\frac{t}{2\pi}\right)^{c+it-1/2} \\ &\quad \times \left(\frac{t+U}{2\pi}\right)^{(2c+2it+2iU-1)/4} e^{-(3it+iU)/2} dt \\ &\quad + O\left(\sum_{n > N_0} |h(n, U)| n^{-c} T^{(6c-3)/4}\right). \end{aligned}$$

The expression in the O -term is clearly $\ll_{\varepsilon} T^{3/4+\varepsilon}$, and it remains to estimate

$$(3.8) \quad \sum_{n > N_0} h(n, U) n^{-c} e^{-iU/2} \int_{T/2}^T K(t) e^{if_n(t)} dt,$$

with

$$K(t) := \left(\frac{t}{2\pi}\right)^{c-1/2} \left(\frac{t+U}{2\pi}\right)^{(2c-1)/4} \ll_{\varepsilon} T^{3/4+\varepsilon}$$

for $c = 1 + \varepsilon$. In (3.8) we have

$$\begin{aligned} f_n(t) &:= t \log \frac{t}{2\pi} + \frac{1}{2}(t+U) \log \frac{t+U}{2\pi} - \frac{3}{2}t - t \log n, \\ (3.9) \quad f'_n(t) &= \log \frac{t}{2\pi} + \frac{1}{2} \log \frac{t+U}{2\pi} - \log n, \\ f''_n(t) &= \frac{1}{t} + \frac{1}{2(t+U)} = \frac{3t+2U}{2t(t+U)} \asymp \frac{1}{T}. \end{aligned}$$

In the sum in (3.8) set $n = [N_0] + \nu$, $\nu \in \mathbb{N}$. By the second derivative test (Lemma 2.2 of [5]) the contribution of the term $\nu = 1$ is $\ll T^{\varepsilon-1/4}$. The terms $\nu \geq 2$ are estimated by the use of the first derivative test (Lemma 2.1 of [5]). For $\nu \gg N_0$ it is found that $|f'_n(t)| \gg 1$, while for $\nu \ll N_0$ one has, for a suitable $c > 0$,

$$|f'_n(t)| = \log \frac{[N_0] + \nu}{\frac{t}{2\pi} \sqrt{\frac{t+U}{2\pi}}} \geq \log \left(1 + c\nu T^{-3/2}\right) \gg \nu T^{-3/2}.$$

Thus the contribution of the terms with $\nu \geq 2$ will be

$$\begin{aligned} &\ll_{\varepsilon} T^{3/4+\varepsilon} \left(\sum_{2 \leq \nu \ll N_0} + \sum_{\nu \gg N_0} \right) |h([N_0] + \nu, U)| n^{-c} \max_{T/2 \leq t \leq T} |f'_n(t)|^{-1} \\ &\ll_{\varepsilon} T^{3/4+\varepsilon} \left(\sum_{2 \leq \nu \ll N_0} (N_0 + \nu)^{-1-\varepsilon} T^{3/2} \nu^{-1} + \sum_{\nu \gg N_0} \nu^{\varepsilon/2-1-\varepsilon} \right) \\ &\ll_{\varepsilon} T^{3/4+\varepsilon}, \end{aligned}$$

since $N_0 \asymp T^{3/2}$.

It remains to deal with I_1 . We have

$$\begin{aligned} (3.10) \quad I_1 &= e^{-3\pi i/8} \int_{T/2}^T \sum_{n \leq N_0} h(n, U) n^{-1/2} e^{-iU/2} e^{if_n(t)} dt + O_{\varepsilon}(T^{3/4+\varepsilon}) \\ &= e^{-iU/2} e^{-3\pi i/8} \sum_{n \leq N_0} h(n, U) n^{-1/2} \mathcal{J}_n(T) + O_{\varepsilon}(T^{3/4+\varepsilon}), \end{aligned}$$

say, where

$$(3.11) \quad \mathcal{J}_n(T) := \int_{T/2}^T e^{if_n(t)} dt.$$

The exponential integral in (3.11) has a saddle point t_n , the root of the equation $f'_n(t) = 0$, which occurs for

$$(3.12) \quad t_n^2(t_n + U) = 8\pi^3 n^2.$$

It is immediately seen that $t_n \asymp n^{2/3} \asymp T$, and that the equation (3.12) is cubic in t_n for a given n and U , and thus it is not easily solvable. However, its solution can be found asymptotically by an iterative process. First from (3.12) we have

$$t_n^3 = 8\pi^3 n^2 + O(UT^2),$$

which yields

$$(3.13) \quad t_n = 2\pi n^{2/3} + O(U).$$

As we require $T/2 \leq t_n \leq T$, from (3.12) we infer that $N_1 \leq n \leq N_0$, where N_0 is defined by (3.2) (this is the reason for the choice of N_0) and

$$N_1 := \sqrt{\frac{(T/2)^2(T/2 + U)}{8\pi^3}}.$$

Setting $n = [N_1] - \mu$, $1 \leq \mu < N_1$ and proceeding similarly as in the estimation of the sum with $\nu \geq 2$ after (3.9), we obtain that the contribution of $n \leq N_1$ to I_1 in (3.10) will be $O_\varepsilon(T^{3/4+\varepsilon})$. Since $\mathcal{J}_n(T) \ll T^{1/2}$ (see (3.16) and (3.17)) it transpires that the condition $N_1 \leq n \leq N_0$ may be replaced by $T_1 \leq n \leq T_0$, where T_0, T_1 are defined by (2.3). In this process the total error term will be $O_\varepsilon(T^{3/4+\varepsilon})$. Therefore, for the range $0 < U \leq T^{1/2-\varepsilon}$, we have uniformly

$$(3.14) \quad I_1 = e^{-iU/2} e^{-3\pi i/8} \sum_{T_1 \leq n \leq T_0} h(n, U) n^{-1/2} \mathcal{J}_n(T) + O_\varepsilon(T^{3/4+\varepsilon}),$$

and the range of summation in (3.14) is easier to deal with than the range of summation $N_1 < n \leq N_0$. At this point we shall evaluate $\mathcal{J}_n(T)$ by one of the theorems related to exponential integrals with a saddle point, e.g., by the one on p. 71 of the monograph by Karatsuba–Voronin [13]. This says that, if $f(x) \in C^{(4)}[a, b]$,

$$(3.15) \quad \int_a^b e^{2\pi i f(x)} dx = e^{\pi i/4} \frac{e^{2\pi i f(c)}}{\sqrt{f''(c)}} + O(AV^{-1}) \\ + O\left(\min(|f'(a)|^{-1}, \sqrt{a})\right) + O\left(\min(|f'(b)|^{-1}, \sqrt{b})\right),$$

where

$$0 < b - a \leq V, \quad f'(c) = 0, \quad a \leq c \leq b, \\ f''(x) \asymp A^{-1}, \quad f^{(3)}(x) \ll (AV)^{-1}, \quad f^{(4)}(x) \ll A^{-1}V^{-2}.$$

We shall apply (3.15) with

$$f(x) = \frac{f_n(x)}{2\pi}, \quad c = t_n, \quad a = \frac{1}{2}T, \quad b = T, \quad V = \frac{1}{2}T, \quad f''(x) \asymp \frac{1}{T},$$

so that $A = T, f^{(3)}(x) \ll T^{-2}, f^{(4)}(x) \ll T^{-3}$, which is needed. The term $O(AV^{-1})$ in (3.15) will make a contribution which is $O_\varepsilon(T^{3/4+\varepsilon})$, and the same assertion will hold for the other two error terms in (3.15). This will lead to

$$(3.16) \quad I_1 = e^{-iU/2} e^{-3\pi i/8} \sum_{T_1 \leq n \leq T_0} h(n, U) n^{-1/2} e^{\pi i/4} \frac{e^{if_n(t_n)}}{\sqrt{\frac{f_n''(t_n)}{2\pi}}} + O_\varepsilon(T^{3/4+\varepsilon}).$$

Since, by (3.9) and (3.12),

$$\begin{aligned} e^{if_n(t)} &= \exp\left\{i\left(t_n \log \frac{t_n}{2\pi} + \frac{1}{2}(t_n + U) \log \frac{(t_n + U)}{2\pi} - \frac{3}{2}t_n - t_n \log n\right)\right\} \\ &= \exp\left\{\frac{1}{2}it_n \log \frac{t_n^2(t_n + U)}{8\pi^3} + \frac{1}{2}Ui \log \frac{(t_n + U)}{2\pi} - \frac{3}{2}it_n - it_n \log n\right\} \\ &= \exp\left\{\frac{1}{2}iU \log \frac{(t_n + U)}{2\pi} - \frac{3}{2}it_n\right\} \\ &= e^{-3it_n/2} \left(\frac{t_n + U}{2\pi}\right)^{iU/2}, \end{aligned}$$

it follows that

(3.17)

$$I_1 = O_\varepsilon(T^{3/4+\varepsilon})$$

$$+ \sqrt{2\pi} e^{-iU/2} e^{-\pi i/8} \sum_{T_1 \leq n \leq T_0} h(n, U) n^{-1/2} \sqrt{\frac{2t_n(t_n + U)}{3t_n + 2U}} e^{-3it_n/2} \left(\frac{t_n + U}{2\pi}\right)^{iU/2}.$$

In view of (3.12), (3.13) and $n \asymp T^{3/2}$ we have

$$\begin{aligned} n^{-1/2} \sqrt{\frac{2t_n(t_n + U)}{3t_n + 2U}} &= n^{1/2} \sqrt{\frac{2(2\pi)^3}{t_n(3t_n + 2U)}} \\ &= \sqrt{2(2\pi)^3} n^{1/2} t_n^{-1} \left(3^{-1/2} + O(U/T)\right) \\ &= \sqrt{\frac{2}{3}} (2\pi)^3 n^{1/2} \left((8\pi^3 n^2)^{1/3} + O(U^{-1})\right)^{-1} + O(UT^{-5/4}) \\ &= 2\sqrt{\frac{\pi}{3}} n^{-1/6} + O(UT^{-5/4}). \end{aligned}$$

It remains to evaluate

$$(3.18) \quad \mathcal{A} := \mathcal{A}(n, U) = e^{-iU/2} e^{-3it_n/2} \left(\frac{t_n + U}{2\pi} \right)^{iU/2}.$$

To achieve this we need a more precise expression for t_n than (3.13). Putting (3.13) in (3.12) we first have

$$\begin{aligned} t_n^3 &= 8\pi^3 n^2 - Ut_n^2 = 8\pi^3 n^2 - 4\pi^2 U n^{4/3} + O(n^{2/3} U^2), \\ t_n &= (8\pi^3 n^2 - 4\pi^2 U n^{4/3})^{1/3} \left(1 + O(U^2 n^{-4/3}) \right) \\ &= 2\pi n^{2/3} - \frac{1}{3}U + O(U^2 n^{-2/3}). \end{aligned}$$

This yields the second approximation

$$(3.19) \quad t_n = 2\pi n^{2/3} - \frac{1}{3}U + O(U^2 n^{-2/3}).$$

Inserting (3.19) in (3.12) we shall obtain next

$$(3.20) \quad t_n = 2\pi n^{2/3} - \frac{1}{3}U - \frac{U^2 n^{-2/3}}{18\pi} + O(U^3 n^{-4/3}).$$

In fact, (3.20) is just the beginning of an asymptotic expansion for t_n in which each term is by a factor of $U n^{-2/3}$ of a lower order of magnitude than the preceding one. This can be established by mathematical induction. Then we see that (c 's and d 's are effectively computable constants)

$$\begin{aligned} \log \mathcal{A} &= i \left\{ -\frac{1}{2}U - \frac{3}{2}(2\pi)n^{2/3} + \frac{1}{2}U + \frac{U^2 n^{-2/3}}{12\pi} + c_3 U^3 n^{-4/3} + O(U^4 n^{-6/3}) \right. \\ &\quad \left. + \frac{1}{2}U \log \left(n^{2/3} + \frac{U}{3\pi} - \frac{U^2 n^{-2/3}}{36\pi^2} + O(U^3 n^{-4/3}) \right) \right\} \\ &= -3\pi i n^{2/3} + \frac{1}{3}iU \log n + ic_2 U^2 n^{-2/3} + ic_3 U^3 n^{-4/3} + O(U^4 n^{-6/3}), \end{aligned}$$

and therefore

$$(3.21) \quad \mathcal{A} = n^{iU/3} e^{-3\pi i n^{2/3}} \left\{ 1 + id_2 U^2 n^{-2/3} + id_3 U^3 n^{-4/3} + O(U^4 n^{-6/3}) \right\},$$

and we remark that the last term in curly brackets admits an asymptotic expansion. Since $U \ll T^{1/2-\varepsilon}$ by assumption, $U^2 n^{-2/3} \ll T^{-2\varepsilon}$ and thus each term in the expansion will be by a factor of at least $T^{-2\varepsilon}$ smaller than the preceding one. One sees now why the interval of integration in (2.1) and (2.2) is $[T/2, T]$ and not

the more natural $[0, T]$, because the terms $d_j U^j n^{-2j/3}$ become large if n is small, and $K(n, U)$ is then also large.

Inserting (3.21) (as a full asymptotic expansion) and (3.18) in (3.17), we obtain the assertion of Theorem 1, having in mind that we have shown that

$$|I_2| + |I_3| \ll_\varepsilon T^{3/4+\varepsilon}.$$

To prove Theorem 2 note that, with $S(\alpha, N)$ defined by (2.5), we have

$$\begin{aligned} \int_A^B |S(\alpha, N)|^2 d\alpha &= \int_A^B \left(\sum_{n \asymp N} d_3^2(n) + \sum_{m \neq n \asymp N} d_3(m) d_3(n) e^{i\alpha(m^{2/3} - n^{2/3})} \right) d\alpha \\ &= (B - A) \sum_{n \asymp N} d_3^2(n) + O \left(\sum_{m \neq n \asymp N} \frac{d_3(m) d_3(n)}{|m^{2/3} - n^{2/3}|} \right). \end{aligned} \quad (3.22)$$

Note that the function $d_3(n)$ is multiplicative and $d_3(p) = 3$. Thus for $\Re s > 1$ and p a generic prime we have

$$\begin{aligned} \sum_{n=1}^{\infty} d_3^2(n) n^{-s} &= \prod_p \left(1 + d_3^2(p) p^{-s} + d_3^2(p^2) p^{-2s} + \dots \right) \\ &= \zeta^9(s) \prod_p (1 - p^{-s})^9 \prod_p \left(1 + 9p^{-s} + d_3^2(p^2) p^{-2s} + \dots \right) \\ &= \zeta^9(s) F(s), \end{aligned} \quad (3.23)$$

where $F(s)$ is a Dirichlet series that is absolutely convergent for $\Re s > 1/2$. Thus the series in (3.23) is dominated by $\zeta^9(s)$ and it follows, by a simple convolution argument, that

$$\sum_{n \leq x} d_3^2(n) = C_1 x \log^8 x + O(x \log^7 x) \quad (3.24)$$

for a constant $C_1 > 0$.

Next, by using the trivial inequality

$$|ab| \leq \frac{1}{2}(a^2 + b^2)$$

and (3.24), it is seen that the double sum in (3.22) in the O -term is, by symmetry,

$$\begin{aligned} &\ll N^{1/3} \sum_{m \neq n \asymp N} \frac{d_3^2(m) + d_3^2(n)}{|m - n|} \\ &= 2N^{1/3} \sum_{m \asymp N} d_3^2(m) \sum_{n \neq m, n \asymp N} \frac{1}{|n - m|} \\ &\ll N^{4/3} \log^9 N, \end{aligned}$$

since, for a fixed m such that $m \asymp N$,

$$\sum_{n \neq m, n \asymp N} \frac{1}{|n - m|} \ll \log N.$$

From this estimate and (3.24) it follows that

$$(3.25) \quad \int_A^B |S(\alpha, N)|^2 d\alpha \ll N^{4/3} \log^9 N,$$

which implies the assertion of Theorem 2. A further discussion on the cubic moment of $Z(t)$ is to be found in Chapter 11 of the author's monograph [8].

Remark 6. An interesting problem is to determine the true order of the integral in (3.25).

Remark 7. If the divisor function $d_3(n)$ is not present in the definition (2.5), namely if one considers the sum

$$T(\alpha, N) := \sum_{N < n \leq N' \leq 2N} e^{\alpha i n^{2/3}} \quad (\alpha \neq 0),$$

then the problem is much easier. By using Lemma 1.2 and Lemma 2.1 of [5] it easily follows that

$$T(\alpha, N) \ll |\alpha|^{-1} N^{1/3}.$$

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